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2001 J. Phys. A: Math. Gen. 34 2467

(http://iopscience.iop.org/0305-4470/34/11/335)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 2467–2476 www.iop.org/Journals/ja PII: S0305-4470(01)14568-3

# Symmetry of integrable cellular automaton

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Received 5 June 2000, in final form 17 November 2000

**Abstract**

We study an integrable cellular automaton which is called the box–ball system (BBS). The BBS can be derived directly from the integrable differential-difference equation by either ultradiscretization or crystallization. We clarify the integrable structure and the hidden symmetry of the BBS.

PACS numbers: 0565, 0230, 8717

## 1. Introduction

Cellular automata have received much attention in recent theoretical studies in physics, chemistry and biology through numerical experiments on computers (see, for example, the review [1]). It is known that some cellular automata have *soliton*-like solutions; parity-rule filter automata [2], time-reversible parity-rule filter automata [3], a nonlinear $\mathbb{Z}_2$-valued dynamical system with discrete space and time [4], and the box–ball system (BBS) [5, 6]. In this paper based on our results [7–10] we will focus on the BBS, which is simple but rich in integrable structure.

From the viewpoint of the integrable system, the BBS is directly connected with the differential-difference equation called the Bogoyavlensky lattice (generalized Lotka–Volterra model) [11, 12]. There are two ways to show the connection; the 'ultradiscretization' [13] and the 'crystallization' [7]. The former is based on Hirota's bilinear equation of the soliton theory, and by taking a certain limit thereof we recover the evolution equation of the BBS. This indicates the *classical* integrability of the equation. On the other hand, the latter method comes from the *quantum* integrability. We can map the quantum Bogoyavlensky lattice to the integrable vertex model on the two-dimensional square lattice, and the evolution of the BBS appears as a configuration at zero temperature. As the zero temperature corresponds naively to a $q \to 0$ limit of the quantum groups, the latter method can be formulated by use of the crystal base [14].

This paper is organized as follows. In section 2 we define the BBS introduced in [6]. We review in section 3 the ultradiscretization procedure which relates the BBS to the Bogoyavlensky lattice. In section 4 we introduce the crystallization which appears from the *quantum* integrable structure of the Bogoyavlensky lattice. The evolution of the BBS can be

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identified with a ground state of the two-dimensional vertex model. In section 5 we reformulate a construction of the BBS by use of the crystal base. We consider the explicit soliton solutions in section 6. Section 7 is devoted to the concluding remarks.

## 2. Box–ball system

We define the box–ball system introduced in [6]. The evolution rule for the $M$ species of balls on an infinite chain of boxes is as follows. We suppose that each box can have at most one ball.

(a) Take the leftmost ball-$M$ out of its box, and put it in the first empty box to its right.
(b) Take the new leftmost ball-$M$ (as long as it has not yet been moved at this time step), and move it to the first empty box to its right.
(c) Continue process (b) until every ball-$M$ has been moved.
(d) Continue the same processes (a)–(c) for ball-$(M-1)$, ..., ball-2, ball-1. Steps (a)–(d) represent one time step.
(e) Repeat processes (a)–(d).

For our later convenience, we denote an empty box as '0' and a box occupied by ball-$j$ as '$j$'. Below we give some typical examples of soliton scattering for the $M = 3$ case:

$t = 0$: 033100320000000000
1: 000033103200000000
2: 000000031033200000
3: 000000000310033200

$[331] \times [32] \to [31] \times [332]$

$t = 0$: 031102200000000000
1: 000031122000000000
2: 000000011322000000
3: 000000000110322000

$[311] \times [22] \to [11] \times [322]$

$t = 0$: 032100000220100000000000000
1: 000032100002021000000000000
2: 000000032100200210000000000
3: 000000000032120002100000000
4: 000000000000013220021000000
5: 000000000000001003200221000

$[321] \times \Big([22] \times [1]\Big) \to [1] \times [32] \times [221]$

$t = 0$: 032102200010000000000000000000
1: 000031022201000000000000000000
2: 000000310020221000000000000000
3: 000000003102000221000000000000
4: 000000000031200000221000000000
5: 000000000000132000000221000000
6: 000000000000010320000000221000.

$\Big([321] \times [22]\Big) \times [1] \to [1] \times [32] \times [221].$

Throughout this paper we denote $[c_1 \dots c_p]$ ($c_1 \geqslant \cdots \geqslant c_p > 0$) as a 'soliton' of length $p$. One can easily check that this non-increasing sequence does indeed behave like a soliton, and that it is stable after a collision though the internal degree of freedom of the solitons can change (see the above examples). Note that there exists a phase shift after a collision which depends on the initial condition, and that the out-going state does not depend on the order of the scattering (see the last two examples).

When we set $u^t_{n,j}$ as the number of ball-$j$ (for $j = 1, \ldots, M$) on the $n$th box at time $t$, the evolution equation is written as

$$u^{t+1}_{n,j} = \min\left[v^t_{n,j}, 1 - \sum_{i=1}^{j} u^t_{n,i} - \sum_{i=j+1}^{M} u^{t+1}_{n,i}\right] \tag{2.1a}$$

$$u^t_{n,j} + v^t_{n,j} = u^{t+1}_{n,j} + v^t_{n+1,j}. \tag{2.1b}$$

In (2.1) we set

$$u^t_{n,j} = Y^t_{n,j} + Y^t_{n+1,j+1} - Y^t_{n+1,j} - Y^t_{n,j+1} \tag{2.2}$$

with a condition $Y^t_{n,M+1} = Y^{t-1}_{n,1}$. As will be shown later in (3.4), $Y^t_{n,j}$ is an ultradiscrete limit of the $\tau$-function of the Bogoyavlensky lattice. As a result we obtain

$$Y^{t+1}_{n+1,j} + Y^t_{n,j+1} = \max\left[Y^t_{n,j} + Y^{t+1}_{n+1,j+1}, Y^t_{n+1,j+1} + Y^{t+1}_{n,j} - 1\right] \tag{2.3}$$

which can also be regarded as the ultradiscrete bilinear equation (see (3.3) below).

## 3. Ultradiscretization

The ultradiscretization, which relates the BBS with the integrable differential-difference equation called the Bogoyavlensky lattice [11, 12], was first demonstrated in [13]. The main point is to regard (2.3) as a certain limit of Hirota's bilinear equation.

The Bogoyavlensky lattice is an integrable differential-difference equation [11, 15], and it can be regarded [16] as a discrete analogue of the $(M+1)$-reduced KP equation. The equation of motion is given by

$$\frac{\mathrm{d}V_n}{\mathrm{d}t} = V_n \sum_{j=0}^{M} \left(V_{n+j} - V_{n-j}\right). \tag{3.1}$$

By setting the dynamical variables $V_n$ as $V_n = \tau_{n+M+1}\tau_{n-M}/\tau_{n+1}\tau_n$, the equation of motion (3.1) reduces to

$$D_t \tau_{n+1} \cdot \tau_n = \tau_{n+M+1}\tau_{n-M} - \tau_{n+1}\tau_n. \tag{3.2}$$

Here the operator $D_t$ is the Hirota bilinear operator. A natural discretization of this bilinear equation is [17]

$$\tau^t_n \tau^{t+1-M}_{n-1} - \delta \tau^{t+1}_{n-1}\tau^{t-M}_n = (1-\delta)\tau^{t+1}_n \tau^{t-M}_{n-1} \tag{3.3}$$

where $\delta$ is the unit size of time $t \in \mathbb{Z}$, and the variables $n$ and $t$ are transformed linearly; $t \to Mt - n$, $n \to t$. When we substitute

$$\tau^{Mt+j}_n = \exp\left(\frac{Y^t_{n,j}}{\epsilon}\right) \qquad \delta = -\exp\left(-\frac{1}{\epsilon}\right) \tag{3.4}$$

into (3.3) and take the limit $\epsilon \to +0$, we obtain the ultradiscrete bilinear equation (2.3). A key identity is

$$\lim_{\epsilon \to +0} \epsilon \log(\mathrm{e}^{A/\epsilon} + \mathrm{e}^{B/\epsilon}) = \max(A, B). \tag{3.5}$$

It will become clear later that the parameter $\epsilon$ plays the role of temperature.

## 4. Crystallization I

We explain another method, crystallization [7], to obtain the BBS from the Bogoyavlensky lattice. The Bogoyavlensky lattice (3.1) is integrable both at classical and quantum levels [15]. The classical integrability can be proved by the classical $\boldsymbol{r}$-matrix structure;

$$\{\boldsymbol{L}_n(x) \overset{\otimes}{,} \boldsymbol{L}_m(y)\} = \delta_{m,n}[\boldsymbol{r}(x/y), \boldsymbol{L}_n(x) \otimes \boldsymbol{L}_n(y)]. \tag{4.1}$$

Here the Lax matrix $\boldsymbol{L}_n(x)$ is an $(M+1) \times (M+1)$ matrix,

$$\boldsymbol{L}_n(x) = \begin{pmatrix} xP_n & Q_n & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ Q_n^{-1} & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{4.2}$$

and see [7] for an explicit form of the $\boldsymbol{r}$-matrix. The Poisson algebra is

$$\{P_n, Q_m\} = \delta_{n,m} P_n Q_n \qquad \{P_n, P_m\} = \{Q_n, Q_m\} = 0. \tag{4.3}$$

The dynamical variables of the Bogoyavlensky lattice (3.1) are defined using the local canonical variables $P_n$ and $Q_n$ as

$$V_n = (P_n P_{n+1} \cdots P_{n+M})^{-1} Q_n^{-1} Q_{n+M} \tag{4.4}$$

which satisfies the Poisson algebra [11]

$$\{V_n, V_m\} = 2V_m V_n \sum_{k=1}^{M} (\delta_{m,n+k} - \delta_{m,n-k}). \tag{4.5}$$

We can construct the integrals of motion explicitly from the transfer matrix,

$$t(x) = \mathrm{Tr} \overset{\curvearrowleft}{\prod_n} \boldsymbol{L}_n(x).$$

A benefit of using the local canonical operators becomes clear in quantizing the Bogoyavlensky lattice [15]. By replacing the Poisson algebra (4.3) by the $q$-commutation relation,

$$P_n Q_m = q^{\delta_{n,m}} Q_m P_n \qquad [P_n, P_m] = [Q_n, Q_m] = 0 \tag{4.6}$$

we have the fundamental commutation relation (FCR),

$$\check{\boldsymbol{R}}(x/y; q) \boldsymbol{L}_n(x) \otimes \boldsymbol{L}_n(y) = \boldsymbol{L}_n(y) \otimes \boldsymbol{L}_n(x) \check{\boldsymbol{R}}(x/y; q). \tag{4.7}$$

See [7] for an explicit form of the $\check{\boldsymbol{R}}$-matrix. The quantum Lax matrix $\boldsymbol{L}_n(x)$ in (4.2) defines the quantum Bogoyavlensky lattice [15].

We now give the infinite-dimensional representation for operators $P_n$ and $Q_n$;

$$P_n = \sum_{\ell=-\infty}^{\infty} q^{\ell} |\ell\rangle_n \cdot {}_n\langle \ell| \qquad Q_n = \sum_{\ell=-\infty}^{\infty} |\ell+1\rangle_n \cdot {}_n\langle \ell| \qquad Q_n^{-1} = \sum_{\ell=-\infty}^{\infty} |\ell\rangle_n \cdot {}_n\langle \ell+1|. \tag{4.8}$$

With this representation, we have the vertex model on the square lattice (figure 1) whose Boltzmann weight is determined by the Lax matrix (4.2), and the FCR (4.7) supports the integrability. It should be noted that the vertical line in figure 1 denotes a finite-dimensional

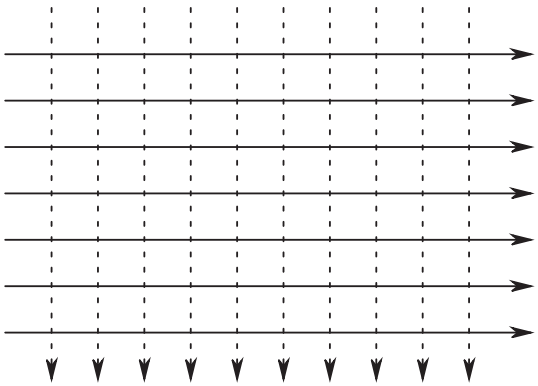


**Figure 1.** The two-dimensional square lattice. Each vertical line is a finite-dimensional space, and plays the role of a box in the BBS. The horizontal lines denote auxiliary spaces, and can be seen as the 'carriers' of the balls.

space. A crucial point in our vertex model is that, with a restriction of the states in the auxiliary space to non-negative integers ($\ell \geqslant 0$ in (4.8)), we have the unique configuration in the *crystal* limit $q \to 0$. This limit is nothing but the zero-temperature limit, and only the ground state survives. It is easy to find that this ground state configuration exactly coincides with the evolution of the BBS. We note that in this realization a state on the top vertical edges corresponds to an initial state of the BBS. See [7] for details.

## 5. Crystallization II

We have shown in the previous section that the BBS can be regarded as the crystal limit of the integrable vertex model, i.e. the configuration of the ground state of the vertex model coincides exactly with the evolution of the BBS. Once we have realized that the $q \to 0$ limit is crucial, we reconstruct the $q \to 0$ vertex model by use of the crystal base [14] for $U_q'(A_M^{(1)})$ from the beginning.

### *5.1. Crystal base for $U_q'(A_M^{(1)})$*

Let $B_k$ be the classical crystal corresponding to the $k$-fold symmetric tensor representation

$$B_k = \left\{ \boxed{m_1 \cdots m_k} \,|\, m_i \in \{1, 2, \ldots, M+1\}, m_1 \leqslant \cdots \leqslant m_k \right\} \tag{5.1}$$

where we have omitted for brevity the $(k-1)$ vertical lines separating the entries. We also denote $b = \boxed{m_1 \cdots m_k} = (x_1, x_2, \ldots, x_{M+1}) \in B_k$ with $x_i = \#\{\ell | m_\ell = i\}$. For each element $b = (x_1, \ldots, x_{M+1})$, we can define the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ (for $i = 1, \ldots, M$) as

$$\tilde{e}_0 b = (x_1 - 1, x_2, \ldots, x_{M+1} + 1) \qquad \tilde{f}_0 b = (x_1 + 1, x_2, \ldots, x_{M+1} - 1)$$
$$\tilde{e}_i b = (x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, x_{M+1}) \qquad \tilde{f}_i b = (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, x_{M+1}).$$

The crystal constructs a coloured oriented graph by defining

$$b \xrightarrow{i} b' \Longleftrightarrow \tilde{f}_i b = b'.$$

We can define the tensor product of the crystal, $B_k \otimes B_\ell = \{b_1 \otimes b_2 | b_1 \in B_k, b_2 \in B_\ell\}$, and the actions of the Kashiwara operators are defined as follows:

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if} \quad \varphi_i(b_1) \geqslant \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if} \quad \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if} \quad \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if} \quad \varphi_i(b_1) \leqslant \varepsilon_i(b_2). \end{cases}$$

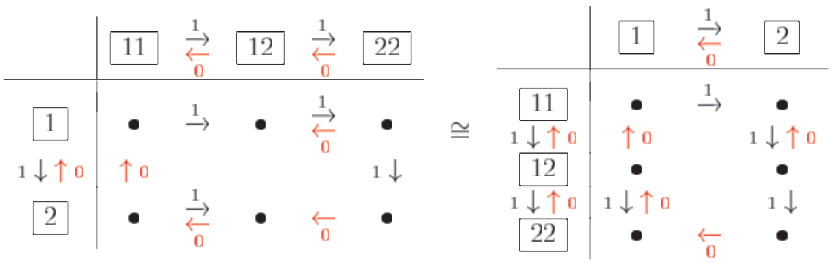
Here we use

$$\varepsilon_i(b) = \max_\ell(\tilde{e}_i^{\,\ell} b \neq 0 | \ell \geqslant 0) \qquad \varphi_i(b) = \max_\ell(\tilde{f}_i^{\,\ell} b \neq 0 | \ell \geqslant 0).$$

In the case of $b = (x_1, \ldots, x_{M+1}) \in B_k$, we have $\varepsilon_0(b) = x_1$, $\varepsilon_i(b) = x_{i+1}$, $\varphi_0(b) = x_{M+1}$, $\varphi_i(b) = x_i$. For the tensor product of the crystal, there is a unique isomorphic map [18],

$$R : B \otimes B' \xrightarrow{\sim} B' \otimes B$$

which commutes with $\tilde{e}_i$ and $\tilde{f}_i$. This map is called the combinatorial $R$ matrix, and is used to calculate a local height probability of the integrable vertex models by the corner transfer matrix method. As a simple example, we show for $M = 1$ an isomorphism between $B_2 \otimes B_1$ and $B_1 \otimes B_2$;




We can define the energy function $H_0$ for the crystal isomorphism. When $b \otimes b' \in B \otimes B'$ is mapped to $\tilde{b}' \otimes \tilde{b} \in B' \otimes B$, the energy function satisfies the following property; for any $j$ such that $\tilde{e}_j(b \otimes b') \neq 0$,

$$H_0(\tilde{e}_j(b \otimes b')) = \begin{cases} H_0(b \otimes b') + 1 & \text{if} \quad j = 0, \varphi_0(b) \geqslant \varepsilon_0(b'), \varphi_0(\tilde{b}') \geqslant \varepsilon_0(\tilde{b}) \\ H_0(b \otimes b') - 1 & \text{if} \quad j = 0, \varphi_0(b) < \varepsilon_0(b'), \varphi_0(\tilde{b}') < \varepsilon_0(\tilde{b}) \\ H_0(b \otimes b') & \text{otherwise.} \end{cases} \tag{5.2a}$$

We further use the generalized energy function $H_i$ (for $i = 1, \ldots, M$); for any $j$ such that $\tilde{f}_j(b \otimes b') \neq 0$,

$$H_i(\tilde{f}_j(b \otimes b')) = \begin{cases} H_i(b \otimes b') + 1 & \text{if} \quad j = i, \varphi_i(b) \leqslant \varepsilon_i(b'), \varphi_i(\tilde{b}') \leqslant \varepsilon_i(\tilde{b}) \\ H_i(b \otimes b') - 1 & \text{if} \quad j = i, \varphi_i(b) > \varepsilon_i(b'), \varphi_i(\tilde{b}') > \varepsilon_i(\tilde{b}) \\ H_i(b \otimes b') & \text{otherwise.} \end{cases} \tag{5.2b}$$

Hereafter we use a notation $\boldsymbol{H}(b \otimes b') = (H_0(b \otimes b'), \ldots, H_M(b \otimes b'))$, and normalize $H_i$ as

$$H_i\Big(\underbrace{\boxed{1 \cdots 1}}_{k} \otimes \underbrace{\boxed{1 \cdots 1}}_{\ell}\Big) = \min(k, \ell).$$

There exists a simple method to give the isomorphism $R : B_k \otimes B_\ell \to B_\ell \otimes B_k$ [19]. We consider $b_1 = (x_1, \ldots, x_{M+1}) \in B_k$ and $b_2 = (y_1, \ldots, y_{M+1}) \in B_\ell$. We represent $b_1 \otimes b_2$ by

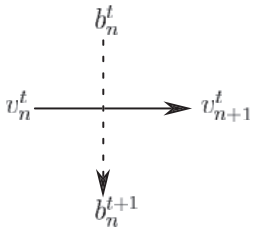
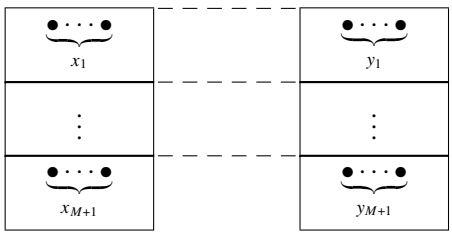



**Figure 2.** We associate $b_n^t$ and $v_n^t$ for each vertex.

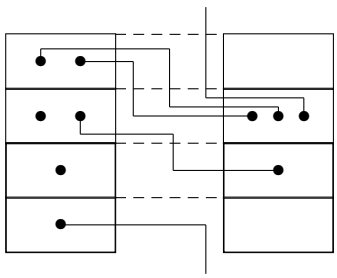
the two column diagrams following the rules given below; each column has $M + 1$ rows, and we put $x_i$ (respectively $y_i$) dots $\bullet$ in the $i$th row of the left (respectively right) column.

$b_1 \otimes b_2 =$




(a) Assume $k \geqslant \ell$ (respectively $k \leqslant \ell$). Pick the dot $\bullet_a$ which is located at the highest position in the right (respectively left) column. Connect it to the dot in the left (respectively right) column which has the lowest (respectively highest) position among all dots whose positions are higher (respectively lower) than that of $\bullet_a$. When there is no such dot in the left (respectively right), return to the bottom (respectively top).
(b) Repeat the previous process for the remaining unconnected dots in the right (respectively left) column.
(c) The crystal isomorphism, $b_1 \otimes b_2 \stackrel{\sim}{\mapsto} b_2' \otimes b_1'$, is obtained by sliding the remaining unpaired dots in the left (respectively right) column to the right (respectively left) one.
(d) The energy function $H_i(b_1 \otimes b_2)$ is given by the number of lines that cross the $(i + 1)$th dotted line.

The example below shows an isomorphism, $\boxed{112234} \otimes \boxed{2223} \simeq \boxed{1124} \otimes \boxed{222233}$, for $M = 3$, and the energy function is $\boldsymbol{H} = (1, 3, 1, 0)$.



### *5.2. Cellular automaton*

As an analogy with the construction of the BBS in section 4 we can define the evolution operator from the combinatorial $R$-matrix. We consider the two-dimensional lattice (figure 1),

and assign $B_1$ and $B_\kappa$ for the vertical dotted lines and the horizontal full lines, respectively. We then associate the combinatorial $R$-matrix, $R : v_n^t \otimes b_n^t \stackrel{\sim}{\to} b_n^{t+1} \otimes v_{n+1}^t$, for each vertex (figure 2). Correspondingly, the evolution operator $T_\kappa$ is given by

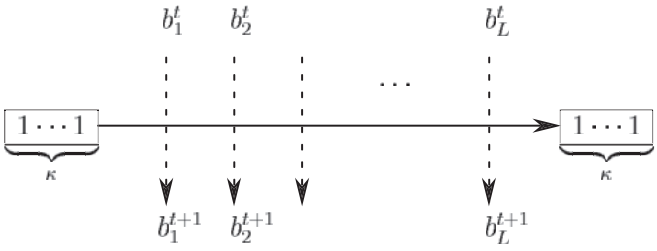



**Figure 3.** Evolution operator $T$.

$$T_\kappa : b_1^t \otimes b_2^t \otimes \cdots \otimes b_L^t \longmapsto b_1^{t+1} \otimes b_2^{t+1} \otimes \cdots \otimes b_L^{t+1} \tag{5.3}$$

where we have the crystal isomorphism (figure 3),

$$\underbrace{\boxed{1\cdots 1}}_{\kappa} \otimes b_1^t \otimes \cdots \otimes b_L^t \stackrel{\sim}{\longmapsto} b_1^{t+1} \otimes \cdots \otimes b_L^{t+1} \otimes \underbrace{\boxed{1\cdots 1}}_{\kappa}.$$

Here we assume $L \gg 1$ and $b_n^t = \boxed{1}$ for large $n$. If we set $b_n^t = (u_{n,0}^t, u_{n,1}^t, \ldots, u_{n,M}^t)$ and $v_n^t = (v_{n,0}^t, \ldots, v_{n,M}^t)$, we find from an isomorphic rule that the evolution equation can be written as

$$\begin{aligned} u_{n,j}^t - v_{n,j}^t &= \max[X_1 - 1, X_2 - 1, \ldots, X_{j-1} - 1, X_j - \kappa, \ldots, X_M - \kappa, 0] \\ &\quad - \max[X_1 - 1, X_2 - 1, \ldots, X_j - 1, X_{j+1} - \kappa, \ldots, X_M - \kappa, 0] \qquad (5.4a) \\ u_{n,j}^t + v_{n,j}^t &= u_{n,j}^{t+1} + v_{n+1,j}^t \qquad (5.4b) \end{aligned}$$

where $X_\ell = \sum_{i=\ell}^M u_{n,i}^t + \sum_{i=1}^\ell v_{n,i}^t$. We stress that the parameter $\kappa$ plays the role of the capacity of the 'carrier' of balls [20]. By taking the limit $\kappa \to \infty$, we see that the above evolution equation coincides with (2.1).

Another interesting property of the BBS is that the soliton scattering can be identified with the crystal isomorphism [10, 21, 22]; the two-soliton scattering of the BBS

$$[a_1 \cdots a_\ell] \times [c_1 \cdots c_m] \to [c_1' \cdots c_m'] \times [a_1' \cdots a_\ell']$$

coincides with the crystal isomorphism of $U_q(A_{M-1}^{(1)})$;

$$\begin{array}{ccc} B_\ell \otimes B_m & \stackrel{\sim}{\longrightarrow} & B_m \otimes B_\ell \\ \boxed{a_\ell \cdots a_1} \otimes \boxed{c_m \cdots c_1} & \stackrel{\sim}{\longmapsto} & \boxed{c_m' \cdots c_1'} \otimes \boxed{a_\ell' \cdots a_1'} \end{array}.$$

Check the examples in section 2. This can be proved by use of the crystal base theory (see [10, 21] for a proof).

## 6. Soliton solutions

As the ultradiscretization connects the evolution equation (2.3) with the Hirota equation (3.3), the soliton solutions can be given essentially as a reduction of those of the Hirota equation [10, 23]. We can obtain most of the soliton solutions by that process, but some of them are

still missing even in the case of the two-soliton solution. Based on results in [10, 23] and a numerical computation, we give an explicit form of soliton solutions of the ultradiscretized equation (2.3). We can construct $N$-soliton solutions by use of this method.

(a) *One soliton.* We consider a propagation of the one-soliton solution $[c_1 \cdots c_p]$, which corresponds to an element of the $U'_q(A^{(1)}_{M-1})$ crystal, $\boxed{c_p \cdots c_1} = (x_1, \ldots, x_M) \in B_p$. A solution of the bilinear (2.3) is given by

$$Y^t_{n,j} = \max\left[0, C + n - pt - \sum_{i=j}^{M} x_i\right] \tag{6.1}$$

where $C$ is arbitrary.

(b) *Two solitons.* The scattering of two solitons is given by the $U'_q(A^{(1)}_{M-1})$ crystal isomorphism,

$$b_1 \otimes b_2 \simeq b'_2 \otimes b'_1 \tag{6.2}$$

where $b_r, b'_r \in B_{p(r)}$, and we assume $p(1) \geqslant p(2)$. Then the $\tau$-function as a solution of (2.3) is given by

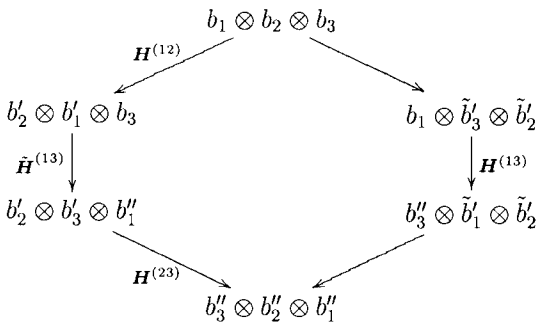
$$Y^t_{n,j} = \max\left[0, \xi^t_{n,j}(1), \xi^t_{n,j}(2), \xi^t_{n,j}(1) + \xi^t_{n,j}(2) - S_j\right] \tag{6.3}$$

$$\xi^t_{n,j}(r) = C(r) + n - p(r)t - \sum_{i=j}^{M} x_i(r)$$

$$S_j = p(2) + H_{j-1}$$

where $C(r)$ are arbitrary, and $b_1 = (x_1(1), \ldots, x_M(1))$, $b'_2 = (x_1(2), \ldots, x_M(2))$. The energy function $H_j$ is defined for the isomorphism (6.2). We should stress that $x_i(2)$ depends on the out-going state.

(c) *Three solitons.* We suppose $p(1) \geqslant p(2) \geqslant p(3)$, and consider a soliton solution whose in-coming state ($t \to -\infty$) corresponds to $b_1 \otimes b_2 \otimes b_3$ with $b_i \in B_{p(i)}$. The out-going state $b''_3 \otimes b''_2 \otimes b''_1$ is constructed by the crystal isomorphism;




Here the arrows denote the crystal isomorphism, and we have labelled the energy function for each isomorphism. The $\tau$-function is given by

$$Y^t_{n,j} = \max_{J \subset \{1,2,3\}} \left[\sum_{r \in J} \xi^t_{n,j}(r) - S^J_j\right] \tag{6.4}$$

where $b_1 = (x_1(1), \ldots, x_M(1))$, $b'_2 = (x_1(2), \ldots, x_M(2))$, $b''_3 = (x_1(3), \ldots, x_M(3))$, and

$$\xi^t_{n,j}(r) = C(r) + n - p(r)t - \sum_{i=j}^{M} x_i(r)$$

$$S_j^{(r_1 r_2)} = \min[p(r_1),\, p(r_2)] + H_{j-1}^{(r_1 r_2)}$$

$$S_j^{(123)} = \sum_{a=1}^{3} p(a) + H_{j-1}^{(12)} + H_{j-1}^{(23)} + \tilde{H}_{j-1}^{(13)}.$$

## 7. Concluding remarks

We have clarified a novel relationship between the box–ball system and the integrable differential-difference equations. The evolution equation is merely an ultradiscrete limit of the Hirota equation, but the BBS has a rich algebraic structure which can be formulated by use of the crystal base. Crystal base theory enables us to generalize the BBS to (a) the BBS with an arbitrary capacity for each box [10] and (b) the BBS associated with other Lie algebras [9, 24]. We have further demonstrated that the ultradiscretized $\tau$-function includes an energy function, but it still remains a future problem as to how to formulate the bilinear equation (2.3) by use of the crystal base operators.

## Acknowledgments

The authors would like to thank T Tokihiro for stimulating discussions. KH thanks the organizers of the workshop 'Mathematical Methods of Regular Dynamics' at the University of Leeds. He also thanks the Yoshida Foundation for Science and Technology for financial support. RI is a Research Fellow of the Japan Society for the Promotion of Science.

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